Nested Multisets, Hereditary Multisets, and Syntactic Ordinals in Isabelle/HOL

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Motivation

- Jasmin needs ordinals for the transfinite Knuth-Bendix ordering
- Dmitriy wants nested multiset ordering

Multisets

Nested Multisets

Hereditary Multisets

Syntactic Ordinals

Signed Hereditary Multisets

Multisets

A non-empty set

```
typedef 'a multiset = {f :: 'a ⇒ nat. finite {x. f x > 0}}
```

Values are constructed by `{#}` **and** `add_mset`

A non-empty set

```
typedef 'a multiset = {f :: 'a ⇒ nat. finite {x. f x > 0}}
```

Values are constructed by `{#}` **and** `add_mset`

@Isabelle User: please use and extend
`$AFP/Nested_Multisets_Ordinals/Multiset_More`
(we slowly move the theorems to the distribution)

Cancellation Simprocs

- Simplify `add_mset a A + F = F + add_mset b (add_mset a B)`

  into `A = add_mset b B`

- Based on the simproc for natural numbers to handle

  `replicate_mset n a = {a} + {a} + ... + {a}`

  (n times)

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Nested Multisets

type variable

type constructor

```
datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
```

keyword to define datatypes and derive theorems

constructors

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```
datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
```

Recursion allowed through bounded natural functor

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Bounded Natural Functors

```
datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
```

- BNF: `('a nmultiset, map_nmultiset, set_nmultiset,` $\aleph_0$`)`

type — **functorial action** — **natural transformation** — **upper bound on cardinality**

Bounded Natural Functors

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datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
```

- BNF: `('a nmultiset, map_nmultiset, set_nmultiset,` $\aleph_0$`)`

  - `'a nmultiset` ← **type**
  - `map_nmultiset` ← **functorial action**
  - `set_nmultiset` ← **natural transformation**
  - $\aleph_0$ ← **upper bound on cardinality**

- BNFs are closed under composition

Bounded Natural Functors

```
datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
```

- BNF: `('a nmultiset, map_nmultiset, set_nmultiset,` $\aleph_0$`)`

  type — **functorial action** — **natural transformation** — **upper bound on cardinality**

- BNFs are closed under composition
- Datatypes and codatatypes are BNFs

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Bounded Natural Functors

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datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
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- BNF: `('a nmultiset, map_nmultiset, set_nmultiset,` $\aleph_0$`)`

  type — **functorial action** — **natural transformation** — **upper bound on cardinality**

- BNFs are closed under composition
- Datatypes and codatatypes are BNFs
- Some non-datatypes are also be BNFs, e.g., multisets

```
datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
```

Induction principle:

```
⋀x. P (Elem x)
⋀NM. (⋀N. N ∈ set_multiset NM ⟹ P N) ⟹ P (MSet NM)
P N
```

```
datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
```

Induction principle:

$$\frac{\bigwedge x.\ P\ (\mathsf{Elem}\ x) \qquad \bigwedge NM.\ (\bigwedge N.\ N \in \mathsf{set\_multiset}\ NM \Longrightarrow P\ N) \Longrightarrow P\ (\mathsf{MSet}\ NM)}{P\ N}$$

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```
datatype 'a nmultiset =
    Elem 'a | MSet (('a nmultiset) multiset)
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Induction principle:

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Allows to define recursive functions:

```
primrec depth where
  depth (Elem x) = 0
| depth (MSet M) =
    (let X = set (map_mset depth M) in
     if X = {} then 0 else Max X + 1)
```

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```
datatype 'a nmultiset =
  Elem 'a | MSet (('a nmultiset) multiset)
```

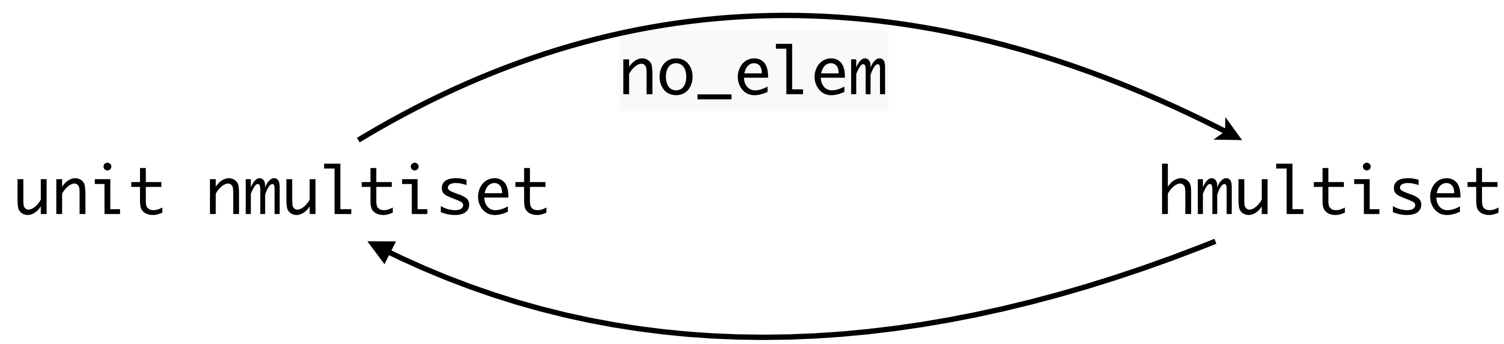
We often don't want `Elem`. Three options:

1. `'a = ∅`
   - `'a` cannot be empty in Isabelle

2. ```
   datatype hmultiset =
     HMSet (hmultiset multiset)
   ```
   - generates selector, induction principle, recursion scheme

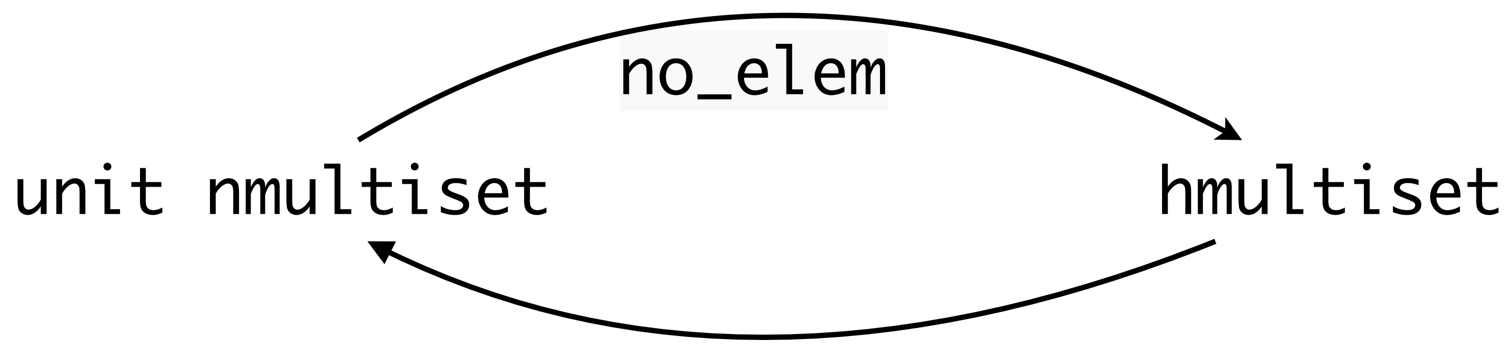
3. `typedef` **and** `no_elem` **predicate**
   - allows to lift definition via the Lifting tool
   ```

```
Abs_hmultiset (MSet M) = HMSet (map_mset Abs_hmultiset M)
```




```
Rep_hmultiset (HMSet M) = MSet (map_mset Rep_hmultiset M)
```

```
Abs_hmultiset (MSet M) = HMSet (map_mset Abs_hmultiset M)
```




```
Rep_hmultiset (HMSet M) = MSet (map_mset Rep_hmultiset M)
```

Lift definition via the Lifting and Transfer tool, e.g.:

```
A < B ⟷ Rep_hmultiset A < Rep_hmultiset B
```

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Syntactic Ordinals

Cantor normal form for the ordinals below ϵ_0

$$\alpha ::= \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_n} \cdot c_n$$

$$\text{where } c_i \in \mathbb{N}^{>0} \text{ and } \alpha_1 > \cdots > \alpha_n$$

$$\alpha ::= \{ \underbrace{\alpha_1, \ldots, \alpha_1}_{c_1 \text{ occurrences}}, \ldots, \underbrace{\alpha_n, \ldots, \alpha_n}_{c_n \text{ occurrences}} \}$$

E.g.:

$$\{\} = 0 \qquad \{0\} = \{\{\}\} = \omega^0 = 1$$

$$\{1\} = \{\{\{\}\}\} = \omega^{\omega^0} = \omega \qquad \{\omega\} = \omega^\omega$$

Hessenberg addition [Ludwig and Waldmann]:

Definition 7 (Hessenberg Addition). Let $\oplus\colon \mathbf{O} \times \mathbf{O} \to \mathbf{O}$ be the following function:

– For $\alpha \in \mathbf{O} \setminus \{0\}$ we define:

$$0 \oplus 0 = 0$$
$$0 \oplus \alpha = \alpha$$
$$\alpha \oplus 0 = \alpha$$

– Let for natural numbers $m, m' \in \mathbb{N}^{>0}$, $n_1, \ldots, n_m$, $n'_1, \ldots, n'_{m'} \in \mathbb{N}^{>0}$, ordinals $b_1, \ldots, b_m$, $b'_1, \ldots, b'_{m'} \in \mathbf{O}$ such that $b_1 > b_2 > \cdots > b_m$ and $b'_1 > b'_2 > \cdots > b'_{m'}$,

$$\alpha = \sum_{i=1}^{m} (\omega^{b_i} \cdot n_i),\ \beta = \sum_{i=1}^{m'} (\omega^{b'_i} \cdot n'_i) \in \mathbf{O}$$

Isabelle:

```
A + B = HMSet (hmsetmset A + hmsetmset B)
```

Hessenberg multiplication [Ludwig and Waldmann]:

Definition 8 (Hessenberg Multiplication). Let $\odot \colon \mathbf{O} \times \mathbf{O} \to \mathbf{O}$ be the following function:

- For $\alpha \in \mathbf{O} \setminus \{0\}$ we define:

$$0 \odot 0 = 0$$
$$0 \odot \alpha = 0$$
$$\alpha \odot 0 = 0$$

- Let for $m, m' \in \mathbb{N}^{>0}, n_1, \dots, n_m, n'_1, \dots, n'_{m'} \in \mathbb{N}^{>0}, b_1, \dots, b_m, b'_1, \dots, b'_{m'} \in \mathbf{O}$ such that $b_1 > b_2 > \dots > b_m$ and $b'_1 > b'_2 > \dots > b'_{m'}$,

$$\alpha = \sum_{i=1}^{m} (\omega^{b_i} \cdot n_i),\ \beta = \sum_{j=1}^{m'} (\omega^{b'_j} \cdot n'_j)$$

We define then

$$\alpha \odot \beta = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m'} \left(\omega^{b_i \oplus b'_j} \cdot (\mathrm{coeff}(\alpha,\, b_i) \cdot \mathrm{coeff}(\beta,\, b'_j))\right)$$

Isabelle:

```
A * B = HMSet (map_mset (case_prod (op +))
                   (hmsetmset A ×# hmsetmset B))
```

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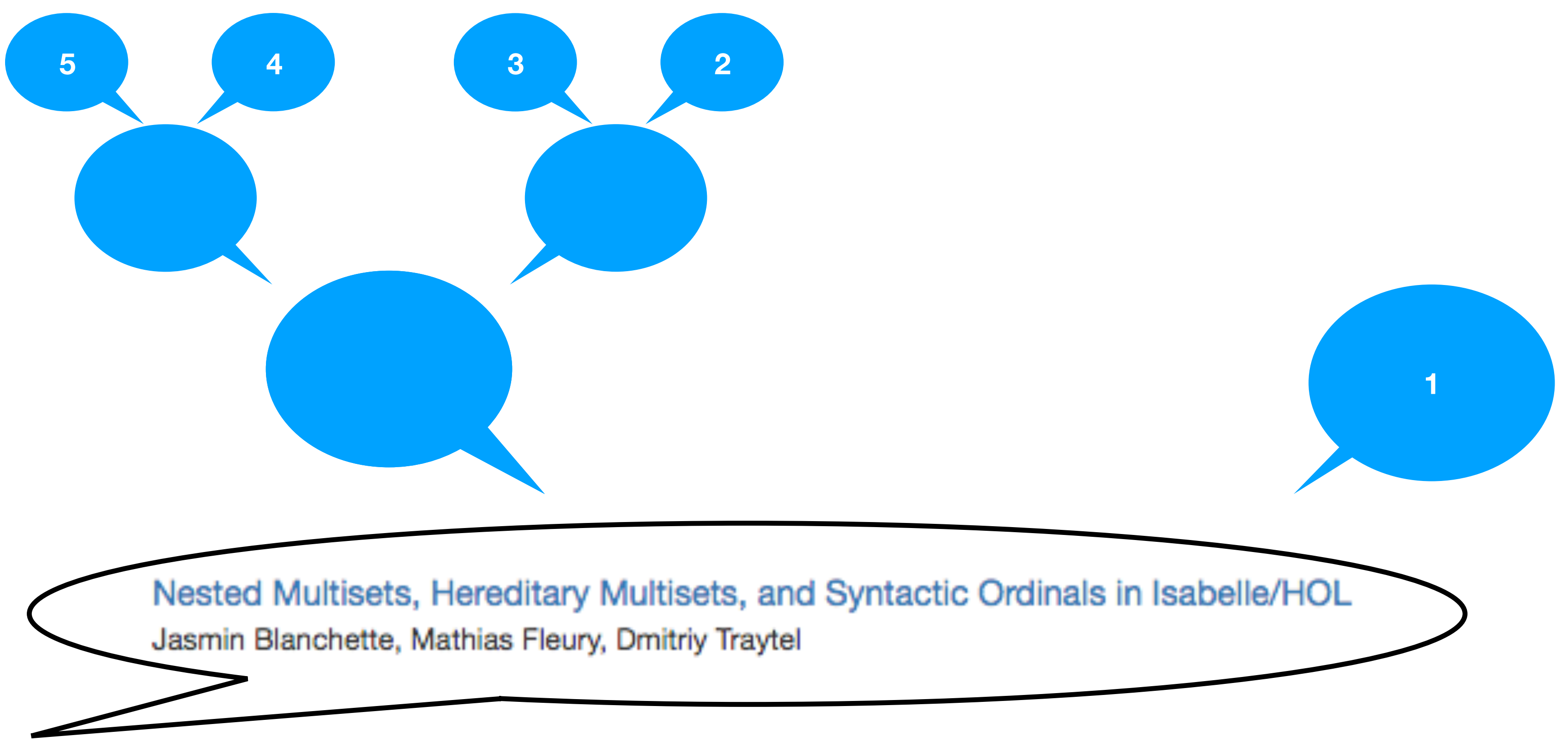
The “Question” Hydra



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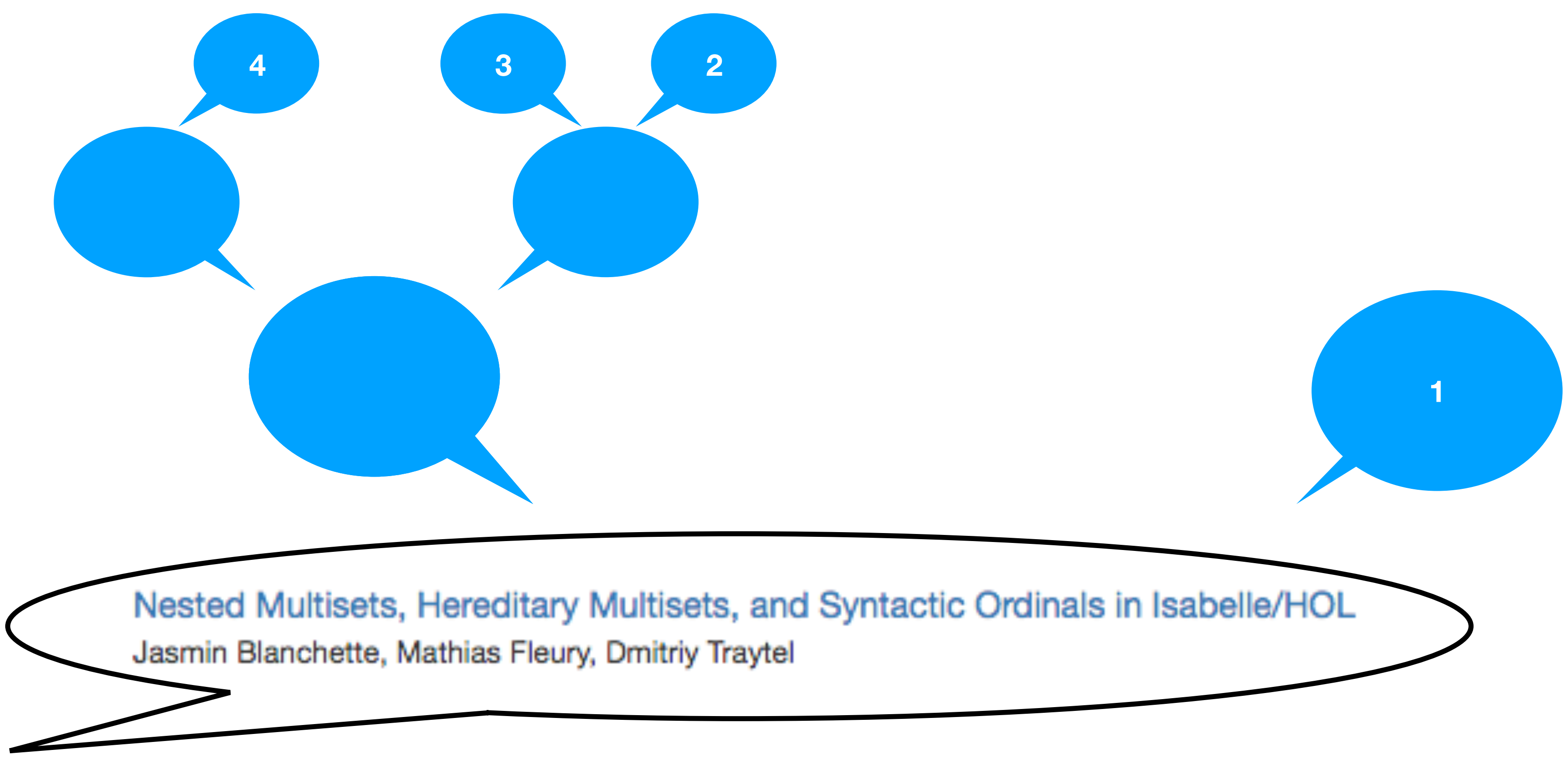

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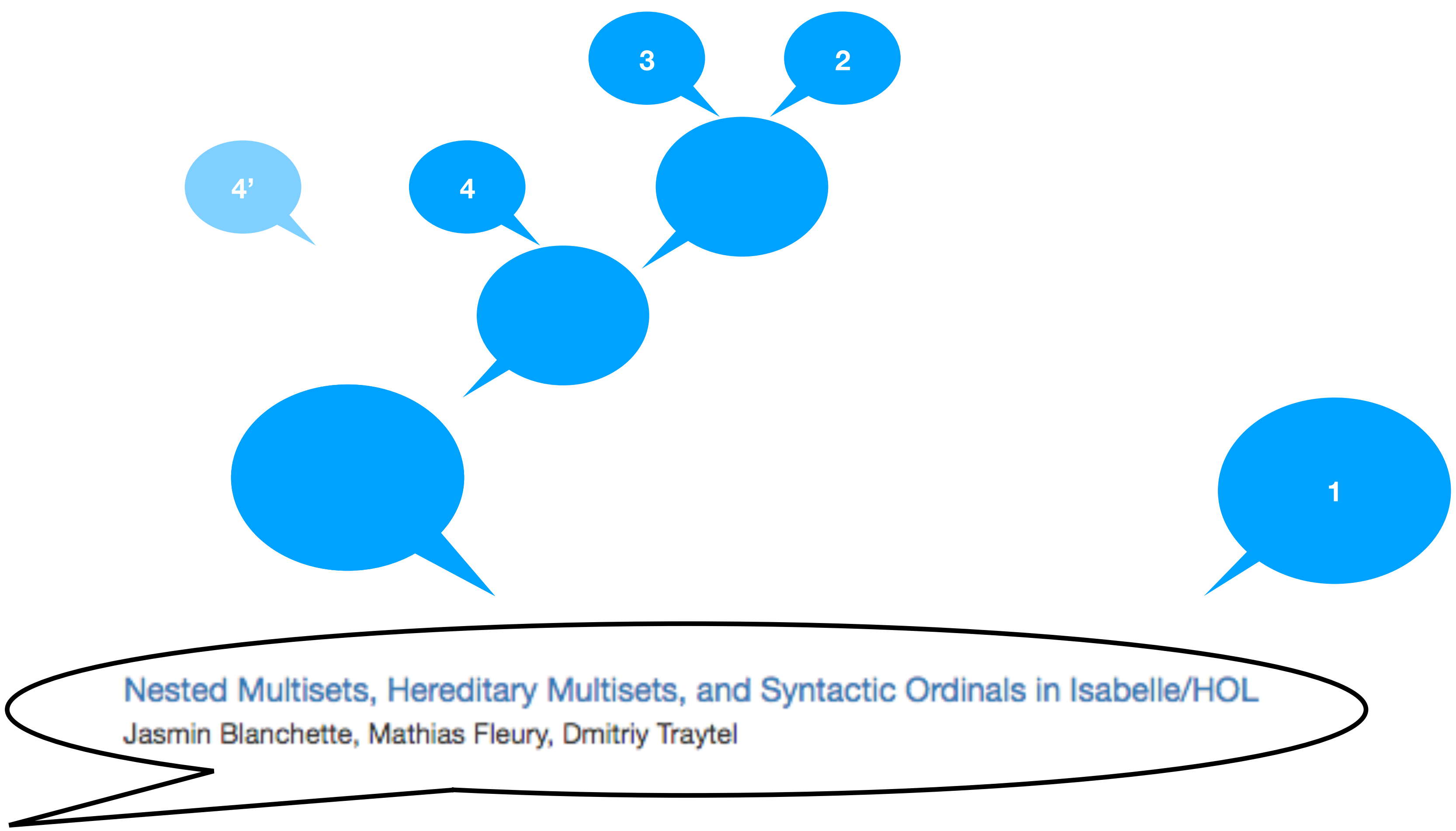

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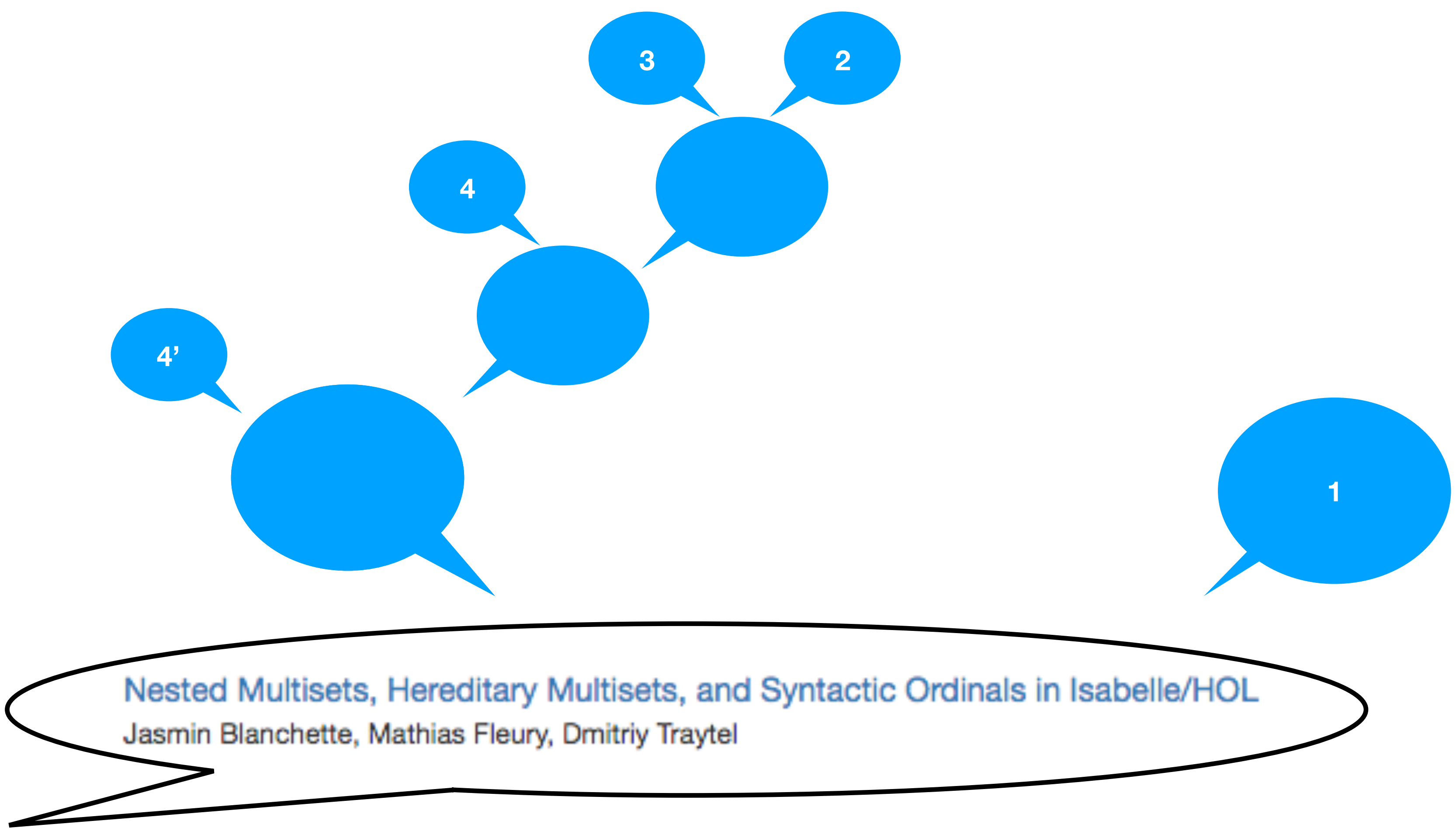

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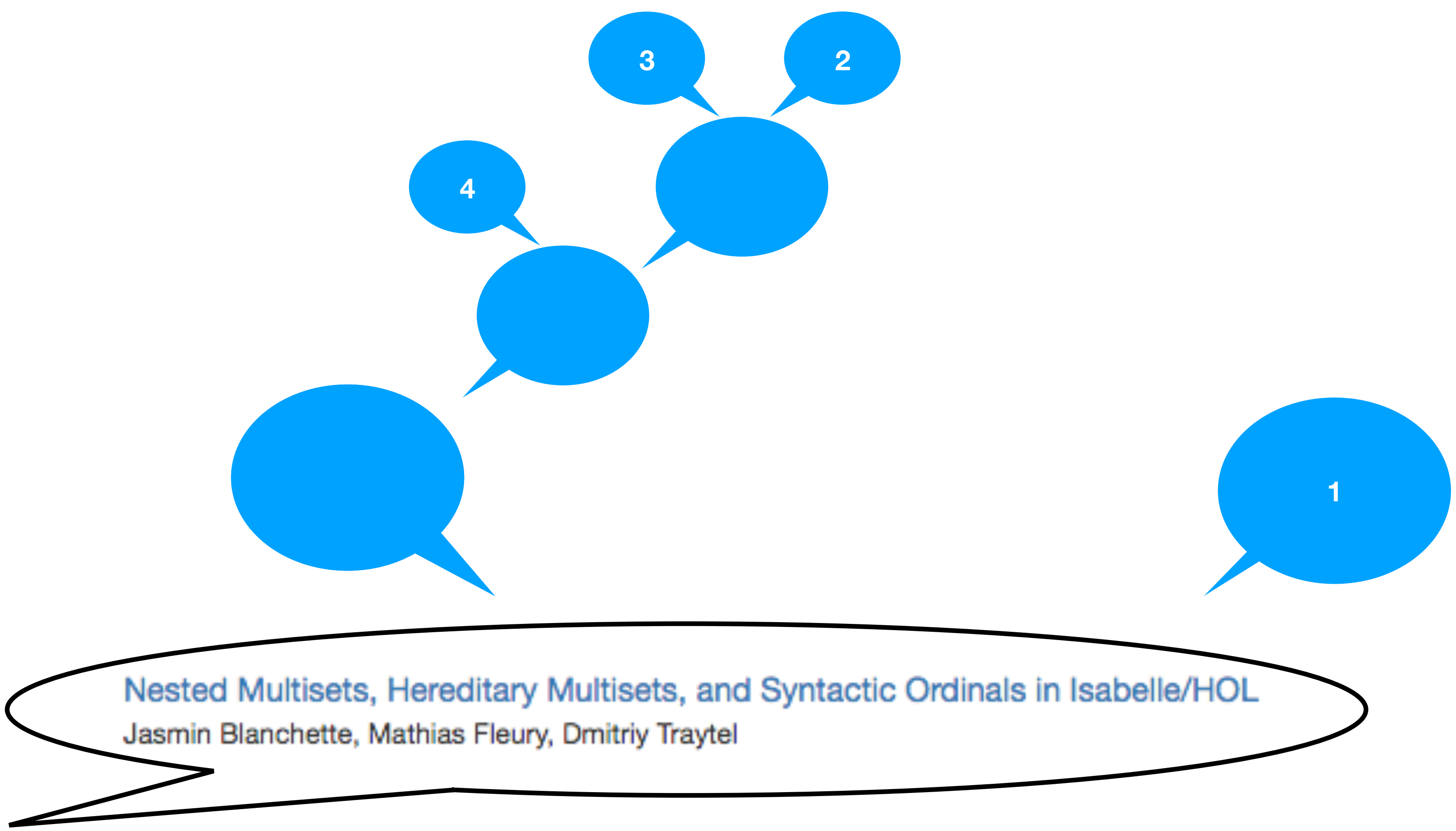

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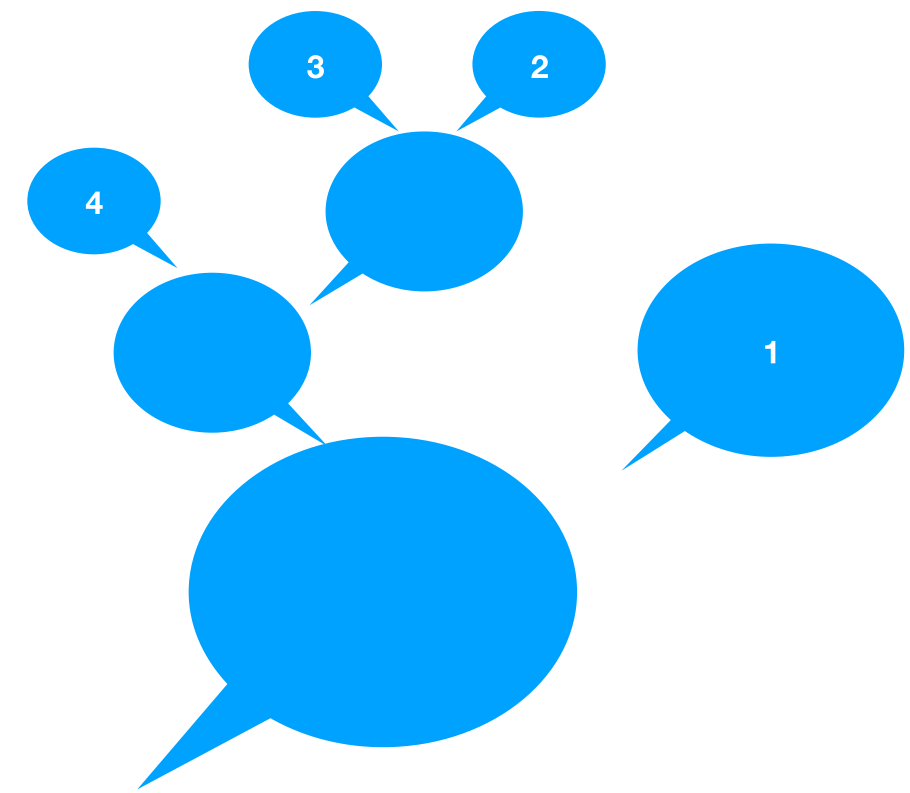

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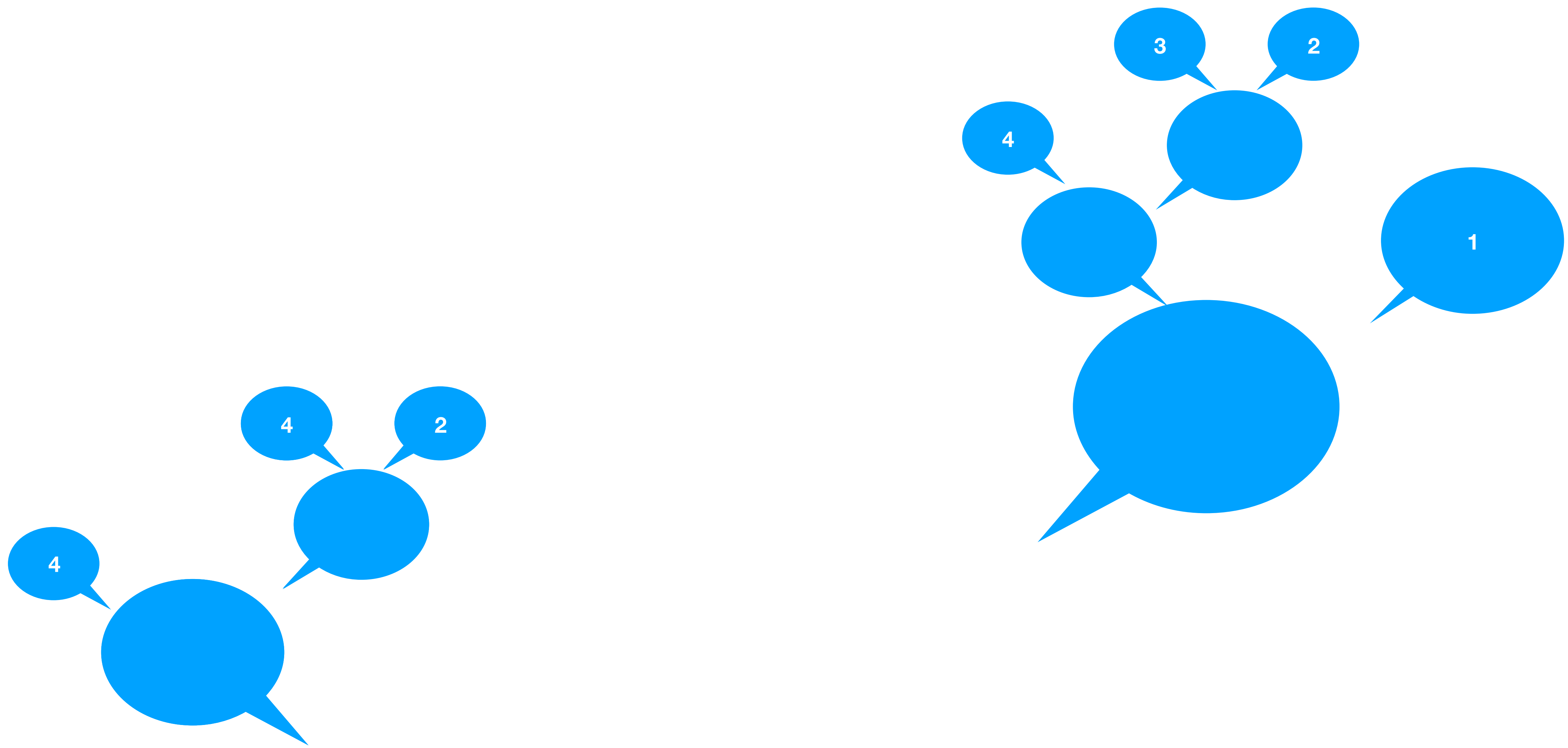

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Will there be a coffee break?

encode(   ) = 0

encode() = ω^encode() + encode()

Will there be a coffee break?

```
encode(        )  =  0

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```

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                  = ω^(ω^encode(   ) + encode(   )
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```

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encode(              )  = ω^encode(   ) + encode(   )

                        = ω^(ω^encode(   ) + encode(   )

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                         = ω^(ω^encode(   ) + encode(   )

                               + encode(   )

encode(              )   = ω^encode(   ) + encode(        )

                         = 2 * ω^encode(   ) + encode(   )
```

Will there be a coffee break?

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encode(        )  =  0

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```
encode(        )  = ω^encode(   ) + encode(   )
                  = ω^(ω^encode(   ) + encode(   )
                      + encode(   )
      V
encode(        )  = ω^encode(   ) + encode(   )
                  = 2 * ω^encode(   ) + encode(   )
```

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```
lemma (* [Ludwig and Waldmann] *)
  assumes α2 + β2 * γ < α1 + β1 * γ and β2 ≤ β1 and γ < δ
  shows α2 + β2 * δ < α1 + β1 * δ
```

Sketch of the “ideal” proof

have $\beta 2 * (\delta - \gamma) < \beta 1 * (\delta - \gamma)$

thus the result

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```
lemma (* [Ludwig and Waldmann] *)
  assumes α2 + β2 * γ < α1 + β1 * γ and β2 ≤ β1 and γ < δ
  shows α2 + β2 * δ < α1 + β1 * δ
```

Sketch of the “ideal” proof

have β2 * (δ - γ) < β1 * (δ - γ)

thus the result

But: subtraction is ill-behaved

$$\alpha \cdot (\beta - \gamma) = \omega^2 + \omega \quad \neq \quad \omega = \alpha \cdot \beta - \alpha \cdot \gamma$$

$$\text{where } \alpha = \omega^2 + \omega \text{ and } \qquad \beta = 1 \text{ and } \gamma = \omega$$

Lemma: $\alpha_1 + \beta_1 \gamma > \alpha_2 + \beta_2 \gamma$

$\beta_1 \geq \beta_2$

$\delta > \gamma$

$\Rightarrow \alpha_1 + \beta_1 \delta > \alpha_2 + \beta_2 \delta$

Proof:

$\beta_1 = \beta_0 + \beta_1'$, $\beta_2 = \beta_0 + \beta_2'$, $deg(\beta_1') > deg(\beta_2')$ or $\beta_1' = \beta_2' = 0$

$\gamma = \eta + \gamma'$, $\delta = \eta + \delta'$, $deg(\delta') > deg(\gamma')$

$$\alpha_1 + \beta_0 \gamma + \beta_1' \gamma = \alpha_1 + \beta_1 \gamma > \alpha_2 + \beta_2 \gamma = \alpha_2 + \beta_0 \gamma + \beta_2' \gamma$$

(*) $\Rightarrow \alpha_1 + \beta_1' \gamma > \alpha_2 + \beta_2' \gamma$ (by tot., mon.)

$$
\begin{aligned}
\alpha_2 + \beta_2 \delta &= \alpha_2 + \beta_0 \delta + \beta_2' \delta \\
&= \alpha_2 + \beta_0 \delta + \beta_2' \eta + \beta_2' \delta' \\
&\leq \alpha_2 + \beta_0 \delta + \underline{\beta_2' \eta} + \beta_2' \delta' + \underline{\beta_2' \gamma'} \quad \text{(mon.)} \\
&= \underline{\alpha_2 + \beta_2' \gamma} + \beta_0 \delta + \beta_2' \delta' \\
&< \alpha_1 + \underline{\beta_1' \gamma} + \underline{\beta_0 \delta} + \beta_2' \delta' \quad (*, \text{ mon.}) \\
&= \alpha_1 + \beta_1' \eta + \underline{\beta_1' \gamma'} + \beta_0 \eta + \beta_0 \delta' + \underline{\beta_2' \delta'} \\
&\leq \alpha_1 + \beta_1' \eta + \beta_0 \eta + \beta_0 \delta' + \beta_1' \delta' \\
&\quad (deg(\beta_1' \delta') > deg(\beta_1' \gamma' + \beta_2' \\
&\quad \text{or } \beta_1' \delta' = \beta_1' \gamma' = \beta_2' \delta' = 0 \\
&= \alpha_1 + \beta_1 \delta
\end{aligned}
$$

```
lemma (* [Ludwig and Waldmann] *)
  assumes α2 + β2 * γ < α1 + β1 * γ and β2 ≤ β1 and γ < δ
  shows α2 + β2 * δ < α1 + β1 * δ
proof -
  obtain β0 β2a β1a where β1 = β0 + β1a and β2 = β0 + β2a and
    head_ω β2a < head_ω β1a ∨ β2a = 0 ∧ β1a = 0 by …

  obtain η γa δa where γ = η + γa and δ = η + δa and
    head_ω γa < head_ω δa by …

  have α2 + β0 * γ + β2a * γ = α2 + β2 * γ by …
  also have … < α1 + β1 * γ by …
  also have … = α1 + β0 * γ + β1a * γ by …
  finally have *: α2 + β2a * γ < α1 + β1a * γ by …

  have α2 + β2 * δ = α2 + β0 * δ + β2a * δ by …
  also have … = α2 + β0 * δ + β2a * η + β2a * δa by …
  also have … ≤ α2 + β0 * δ + β2a * η + β2a * δa + β2a * γa by …
  also have … = α2 + β2a * γ + β0 * δ + β2a * δa by …
  also have … < α1 + β1a * γ + β0 * δ + β2a * δa by …
  also have … = α1 + β1a * η + β1a * γa + β0 * η + β0 * δa + β2a * δa by …
  also have … ≤ α1 + β1a * η + β0 * η + β0 * δa + β1a * δa by …
  finally show ?thesis by …
qed
```

```
equiv_zmset (Mp, Mn) (Np, Nn) = Mp + Nn = Np + Mn

quotient_type 'a zmultiset =
  'a multiset × 'a multiset / equiv_zmset
```

```
equiv_zmset (Mp, Mn) (Np, Nn) = Mp + Nn = Np + Mn

quotient_type 'a zmultiset =
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equiv_zmset ({}, {7}) ({3}, {3,7})
```

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```
equiv_zmset (Mp, Mn) (Np, Nn) = Mp + Nn = Np + Mn
```

```
quotient_type 'a zmultiset =
  'a multiset × 'a multiset / equiv_zmset
```

```
equiv_zmset ({}, {7}) ({3}, {3,7})
```

Does the operation on pairs respect
the equivalence relation?

```
lift_definition minus_zmultiset
  :: 'a zmultiset ⇒ 'a zmultiset ⇒ 'a zmultiset
  is λ(Mp, Mn) (Np, Nn). (Mp + Nn, Mn + Np)
```

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Associativity of multiplication builds down to

```
An * (Bn * Cn + Bp * Cp - (Bn * Cp + Cn * Bp))
   + (Cn * (An * Bp + Bn * Ap - (An * Bn + Ap * Bp))
   + (Ap * (Bn * Cp + Cn * Bp - (Bn * Cn + Bp * Cp))
   + Cp * (An * Bn + Ap * Bp - (An * Bp + Bn * Ap)))) =
 An * (Bn * Cp + Cn * Bp - (Bn * Cn + Bp * Cp))
   + (Cn * (An * Bn + Ap * Bp - (An * Bp + Bn * Ap))
   + (Ap * (Bn * Cn + Bp * Cp - (Bn * Cp + Cn * Bp))
   + Cp * (An * Bp + Bn * Ap - (An * Bn + Ap * Bp))))
```

Associativity of multiplication builds down to

```
An * (Bn * Cn + Bp * Cp - (Bn * Cp + Cn * Bp))
   + (Cn * (An * Bp + Bn * Ap - (An * Bn + Ap * Bp))
   + (Ap * (Bn * Cp + Cn * Bp - (Bn * Cn + Bp * Cp))
   + Cp * (An * Bn + Ap * Bp - (An * Bp + Bn * Ap)))) =
 An * (Bn * Cp + Cn * Bp - (Bn * Cn + Bp * Cp))
   + (Cn * (An * Bn + Ap * Bp - (An * Bp + Bn * Ap))
   + (Ap * (Bn * Cn + Bp * Cp - (Bn * Cp + Cn * Bp))
   + Cp * (An * Bp + Bn * Ap - (An * Bn + Ap * Bp))))
```

Magic truncation lemma:

```
lemma a * (c - b) + a * b = a * (b - c) + a * c
  by (metis distrib_left diff_plus_sym_hmset)
```

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Signed hereditary multisets: `hmultiset zmultiset`

unsigned powers → hmultiset

signed coefficient → zmultiset

e.g. `ω^2 - ω - 1`

```
lemma (* new version of Ludwig and Waldmann’s lemma *)
  assumes α2 + β2 * γ < α1 + β1 * γ and β2 ≤ β1 and γ < δ
  shows α2 + β2 * δ < α1 + β1 * δ
proof -
  let ?z = zhmset_of

  have ?z α2 + ?z β2 * ?z δ <
       ?z α1 + ?z β1 * ?z γ + ?z β2 * (?z δ - ?z γ)
     by …
  also have … ≤ ?z α1 + ?z β1 * ?z γ + ?z β1 * (?z δ - ?z γ)
     by …
  finally show ?thesis by …
qed
```

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```
lemma (* new version of Ludwig and Waldmann's lemma *)
  assumes α2 + β2 * γ < α1 + β1 * γ and β2 ≤ β1 and γ < δ
  shows α2 + β2 * δ < α1 + β1 * δ
proof -
  let ?z = zhmset_of

  have ?z α2 + ?z β2 * ?z δ <
       ?z α1 + ?z β1 * ?z γ + ?z β2 * (?z δ - ?z γ)
     by …
  also have … ≤ ?z α1 + ?z β1 * ?z γ + ?z β1 * (?z δ - ?z γ)
     by …
  finally show ?thesis by …
qed
```

Now Waldmann has a proper theoretical foundation for ordinals with signed coefficients

Conclusion

Many nice tools in Isabelle, especially:

- datatypes
- lifting package
- quotients
- Sledgehammer

Formalisation in the Archive of Formal Proofs, also:

- Goodstein sequence
- key lemma towards decidability of Unary PCF